

ON A DYNAMICAL VERSION OF A THEOREM OF ROSENLICHT

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ABSTRACT. Consider the action of an algebraic group G on an irreducible algebraic variety X all defined over a field k . M. Rosenlicht showed that orbits in general position in X can be separated by rational invariants. We prove a dynamical analogue of this theorem, where G is replaced by a semigroup of dominant rational maps $X \dashrightarrow X$. Our semigroup G is not required to have the structure of an algebraic variety and can be of arbitrary cardinality.

1. INTRODUCTION

The starting point for this paper is the following classical theorem of M. Rosenlicht [Ros56, Theorem 2].

Theorem 1.1. *Consider the action of an algebraic group G on an irreducible algebraic variety X defined over a field k .*

(a) *There exists a G -invariant dense open subset $X_0 \subset X$ and a G -equivariant morphism $\phi: X_0 \rightarrow Z$ (where G acts trivially on Z), with the following property. For any field extension K/k and any K -point $x \in X_0(K)$, the orbit $G \cdot x$ equals the fiber $\phi^{-1}(\phi(x))$.*

(b) *Moreover, the field of invariants $k(X)^G$ is a purely inseparable extension of $\phi^*k(Z)$, and one can choose Z and ϕ so that $\phi^*k(Z) = k(X)^G$ (in characteristic zero, this is automatic).*

In short, for points x, y in general position in X , distinct G -orbits $G \cdot x$ and $G \cdot y$ can be separated by rational G -invariant functions. Note, in particular, that G -orbits in X_0 are closed in X_0 .

The rational map $\phi: X \dashrightarrow Z$, with $k(Z) = k(X)^G$, is unique up to birational isomorphism. It is called *the rational quotient* for the G -action on X . For details and further references on this construction and its applications, see [PV94, Chapter 2].

The purpose of this note is to prove a dynamical version of this result, where the algebraic group G is replaced by a semigroup of dominant rational maps $X \dashrightarrow X$. Note the semigroup G is not required to have the structure of an algebraic variety, and can be of arbitrary cardinality. We will say that a closed (or open) subvariety $Y \subset X$ is g -invariant, for some $g \in G$, if $g(Y - D_g) \subset Y$. Here D_g is the indeterminacy locus of $g: X \dashrightarrow X$. As usual, we will say that Y is G -invariant if it is g -invariant for every $g \in G$.

Theorem 1.2. *Let k be a field, X be an irreducible quasi-projective k -variety, and G be a semigroup of dominant rational maps $X \dashrightarrow X$ defined over k .*

Then there exists a dense open subvariety X_0 , a countable collection of closed G -invariant subvarieties $Y_1, Y_2, \dots \subsetneq X_0$ and a dominant morphism $\phi: X_0 \rightarrow Z$ with the following properties.

(a) $\phi \circ g = \phi$, as rational maps $X \dashrightarrow Z$.

(b) *Let K/k be a field extension and $x, y \in X_0(K)$ be K -points which do not lie in the indeterminacy locus of any $g \in G$, or on Y_i for any $i \geq 1$. Then $\phi(x) = \phi(y)$ if and only if $\overline{G \cdot x} = \overline{G \cdot y}$ in X_K .*

(c) The field of invariants $k(X)^G$ is a purely inseparable extension of $\phi^*k(Z)$. Moreover, one can choose Z and ϕ so that $\phi^*k(Z) = k(X)^G$ (in characteristic zero, this is automatic).

Furthermore, if G is a monoid (i.e., contains the identity morphism $X \rightarrow X$) then

(d) X_0 can be chosen to be g -invariant for every invertible element $g \in G$, and

(e) If $x \in X_0(K)$ is as in part (b), then the fiber $\phi^{-1}(\phi(x))$ of x in X_0 equals $\overline{G \cdot x} \cap X_0$.

In short, for points x, y in very general position in X , distinct orbit closures $\overline{G \cdot x}$ and $\overline{G \cdot y}$ can be separated by rational G -invariant functions. Here, as usual, “very general position” means “in a countable intersection of dense open subsets”. Several remarks are in order.

(1) In the case where $G \xrightarrow{\sim} \mathbb{N}$ is generated by a single dominant rational map $\sigma: X \dashrightarrow X$ and $k = \mathbb{C}$, Theorem 1.2 was proved by E. Amerik and F. Campana [AC08, Theorem 4.1] by using techniques of Kähler geometry. Our proof of Theorem 1.2 is purely algebraic; in particular, it is valid in prime characteristic.

(2) A conjecture of A. Medvedev and Th. Scanlon [MS14, Conjecture 7.14] asserts that in the case where k is algebraically closed of characteristic 0, $G \xrightarrow{\sim} \mathbb{N}$ is generated by a single regular endomorphism $X \rightarrow X$ and $k(X)^G = k$ (i.e., Z is a point), X has a k -point with a dense G -orbit. Over \mathbb{C} the Medvedev-Scanlon conjecture follows from the above-mentioned [AC08, Theorem 4.1]. In the case where k is any algebraically closed uncountable field (of arbitrary characteristic), it was proved by the first author, D. Rogalski and S. Sierra [BRS10, Theorem 1.2]. We will reprove it in a stronger form (for an arbitrary semigroup G) as Corollary 6.1 below.

Over a countable field, the Medvedev-Scanlon conjecture (which was, in turn, motivated by an earlier related conjecture of S.-W. Zhang [Zha06, Conjecture 4.1.6]) remains largely open. It has been established only in a small number of special instances (see, in particular, [MS14, Theorem 7.16] and [BGT, Theorem 1.3]), and no counterexamples are known.

(3) While we impose no restriction on the cardinality of the semigroup G in Theorem 1.2, the situation can be partially reduced to the case, where G is countable in the following sense. In the setting of Theorem 1.2 there always exists a countable subsemigroup H of G such that $k(X)^G = k(X)^H$, and $\overline{G \cdot x} = \overline{H \cdot x}$ for $x \in X$ in very general position and away from the indeterminacy loci of every $g \in G$. For a precise statement, see Corollary 6.2. On the other hand, there may not exist a finitely generated subgroup $H \subset G$ with these properties; see Example 6.3.

(4) If X is not quasi-projective, Theorem 1.2 can still be applied to any quasi-projective dense open subvariety $X' \subset X$. Note however, that replacing X by a dense open subvariety X' may make the condition on $x, y \in X_0(K)$ in part (b) more stringent by enlarging the indeterminacy loci of the elements $g \in G$, which we now view as dominant rational maps $X' \dashrightarrow X'$.

(5) The idea behind our construction of a rational map ϕ which separates G -orbit closures in very general position in X is as follows. Since X is quasi-projective, we may assume that $X \subset \mathbb{P}^n$ for some $n \geq 1$. We then set $\phi(x)$ to be the class of the orbit closure $\overline{G \cdot x} \subset \mathbb{P}^n$ in the Hilbert scheme $\text{Hilb}(n)$ of subvarieties of \mathbb{P}^n . The challenge is, of course, to show that this defines a rational map $\phi: X \dashrightarrow \text{Hilb}(n)$. The “quotient variety” Z will then be defined as the closure of the image of this map in $\text{Hilb}(n)$.

Note that our proof of Theorem 1.2 may be viewed as an enhanced version of Rosenlicht’s proof of Theorem 1.1 in [Ros56], where the “separating map” ϕ sends $x \in X$ to the class of the orbit closure $\overline{G \cdot x}$ in the appropriate Chow variety. Leaving aside the difference between the Hilbert scheme and the Chow variety, we note that, as one may expect, the orbit closure $\overline{G \cdot x}$ varies more predictably with x in the case of algebraic group actions than it does when G is an arbitrary semigroup of dominant endomorphisms $X \dashrightarrow X$. For this reason, the construction

of ϕ requires extra work in the dynamical setting (which is carried out in Section 2), and the resulting map only separates orbit closures in very general position. In the last section we show how our arguments can be modified to yield a proof of Theorem 1.1.

(6) When it comes to separating orbit closures by rational invariants in the dynamical setting of Theorem 1.2, “very general position” is, indeed, the best one can do, even in the simplest case, where $G \xrightarrow{\sim} \mathbb{N}$ is a group generated by a single dominant morphism $\sigma: X \rightarrow X$. Examples 6.4 and 6.5 show that if we replace the countable collection of $\{Y_i, i \geq 1\}$ of proper subvarieties of X by a finite collection, Theorem 1.2 will fail. Note that in Example 6.4, σ is an automorphism.

(7) The problem of algorithmically computing a generating set for $k(X)^G$ (over k) for an action of a linear algebraic group G on a variety X , has received a lot of attention. For an overview and further references, see, e.g., [Kem07, Introduction]. We are not aware of any similar algorithms in the dynamical setting of Theorem 1.2, even when G is generated by a single dominant morphism $X \rightarrow X$.

2. A DENSE SET OF RATIONAL SECTIONS

In this section we will consider the following situation:

$$\begin{array}{ccc} V_x & \subset & V \\ \pi \downarrow & & \downarrow \pi \\ x & \in & X \end{array} \quad \begin{array}{c} \swarrow \quad \searrow \\ \vdots \quad \vdots \quad \cdots \quad \vdots \quad \vdots \\ \swarrow \quad \searrow \end{array} \quad \begin{array}{c} s_\lambda, \lambda \in \Lambda \end{array}$$

Here V and X are k -varieties, X is irreducible, $\pi: V \rightarrow X$ is a k -morphism, and $s_\lambda: X \dashrightarrow V$ is a collection of rational sections $X \dashrightarrow V$, indexed by a set Λ . The fiber $\pi^{-1}(x)$ of a point $x \in X$ will be denoted by V_x . Note that we do not assume that V is irreducible and do not impose any restrictions on the cardinality of Λ .

If K/k is a field extension, it will be convenient for us to denote by $X(K)'$ the collection of K -points of X lying away from the indeterminacy locus of s_λ , for every $\lambda \in \Lambda$. In other words, for $x \in X(K)'$, $s_\lambda(x)$ is defined for every $\lambda \in \Lambda$. Note that if Λ is large enough, $X(K)'$ may be empty for some fields K/k , even if K is algebraically closed. On the other hand, the generic point x_{gen} of X lies in $X(K_{\text{gen}})'$, where $K_{\text{gen}} = k(X)$ is the function field of X .

Proposition 2.1. *Assume that the union of $s_\lambda(X)$ over all $\lambda \in \Lambda$ is dense in V . Then there exists a countable collection $\{Y_i, i \geq 1\}$ of proper subvarieties of X with the following property: For any field extension K/k and $x \in X(K)'$ away from $\bigcup_{i=1}^{\infty} Y_i$, the set*

$$\{s_\lambda(x) \mid \lambda \in \Lambda\}$$

is Zariski dense in the fiber $V_x := \pi^{-1}(x)$.

Proof. We begin with two reductions.

Since $k(V)$ is finitely generated over $k(X)$, there is a closed subvariety $V' \subset X \times \mathbb{A}^n$ and a birational isomorphism $j: V \dashrightarrow V'$ over X . Note that replacing V by V' necessitates the removal of the sections $s_\lambda: X \dashrightarrow V$ whose image lies in the indeterminacy locus of the birational isomorphism j . This will not present a problem for us, since the union of the images of the remaining sections will still be dense in V .

Secondly, by generic flatness (see, e.g., [Eis95, Theorem 14.4]), after replacing X by a dense open subvariety, we may assume that π is flat. Consequently, the Hilbert function of the fiber V_x is the same for every K -point $x \in V_x$ and every field K/k ; see, e.g., [Eis95, Exercise 6.11].

Let us denote this common Hilbert function by $h(d)$. That is, if $I_x \subset K[t_1, \dots, t_n]$ is the ideal of V_x in \mathbb{A}^n , and $I_x[d]$ is the K -vector space of homogeneous polynomials of degree d in I_x , then $\dim_K(I_x[d]) = h(d)$.

With these reductions in place, we are ready to proceed with the proof. Let K/k be a field and $x \in X(K)'$. Denote by W_x the closure of $\{s_\lambda(x) \mid \lambda \in \Lambda\}$ in V . Clearly

$$(2.1) \quad W_x \subset V_x \subset \mathbb{A}_K^n.$$

Let $I(V_x)$ and $I(W_x) \subset K[t_1, \dots, t_n]$ be the ideals of V_x and W_x in $K[t_1, \dots, t_n]$, respectively. We will denote the K -vector space of polynomials in t_1, \dots, t_n of total degree $\leq d$ contained in $I(V_x)$ by $I(V_x)[d]$, and similarly for $I(W_x)[d]$. Note that by the above assumption on flatness, $\dim(I(V_x)[d]) = h(d)$ depends only on d and not on the choice of x .

By (2.1), $I(V_x) \subset I(W_x)$. We claim that for every $d \geq 1$ there exists a proper closed subvariety $Y_d \subset X$ such that $\dim(I(W_x)[d]) = h(d)$ for any field K/k and any $x \in X(K)'$ away from Y_d . If we can prove this claim, then for $x \in X(K)'$ away from $Y_1 \cup Y_2 \cup \dots$,

$$I(W_x)[d] = h(d) = I(V_x)[d]$$

for every $d \geq 1$ and thus $I(W_x) = I(V_x)$, so that $W_x = V_x$, as desired.

To prove the claim, let us fix $d \geq 1$ and denote the distinct monomials in t_1, \dots, t_n of degree $\leq d$ by M_1, \dots, M_l . (Here $l = \binom{n+d}{d}$, but we shall not use this in the sequel.) Let $x_{\text{gen}} \in X(K_{\text{gen}})'$ be the generic point of X , where $K_{\text{gen}} = k(X)$ is the function field. An element $\alpha(t_1, \dots, t_n) = \sum_{i=1}^l \alpha_i M_i \in K_{\text{gen}}[t_1, \dots, t_n]$ lies in $I(V_{x_{\text{gen}}})[d]$ if and only if α vanishes on V . Since the images of the rational sections $s_\lambda: X \dashrightarrow \mathbb{A}^n$ are dense in V , this is equivalent to $\alpha(s_\lambda) = 0$ in $k(X)$, for each $\lambda \in \Lambda$. The latter condition is, in turn, equivalent to requiring $\alpha_1, \dots, \alpha_l \in K_{\text{gen}}$ to satisfy the system

$$(2.2) \quad \begin{cases} \alpha_1 M_1(s_\lambda) + \alpha_2 M_2(s_\lambda) + \dots + \alpha_l M_l(s_\lambda) = 0 \\ \lambda \in \Lambda \end{cases}$$

of linear equations over $K_{\text{gen}} = k(X)$. Thus $h(d) := \dim_{K_{\text{gen}}}(I_{x_{\text{gen}}}[d])$, is the dimension of the solution space to this system. If we denote the matrix of the system (2.2) by

$$(2.3) \quad A_{\text{gen}} := \begin{pmatrix} M_1(s_\lambda) & M_2(s_\lambda) & \dots & M_l(s_\lambda) \\ \lambda \in \Lambda \end{pmatrix},$$

then $h(d) = l - r$, where $r := \text{rank}(A_{\text{gen}})$. Here the columns of A_{gen} are indexed by $1, \dots, l$ and the rows by $\lambda \in \Lambda$. (In particular, if Λ happens to be uncountable, A_{gen} will have uncountably many rows.)

We now define $Y_d \subset X$ by the condition that $\text{rank}(A_{\text{gen}}) < r$. That is, Y_d is the intersection of the closed subvarieties of X cut out by the system of equations

$$\begin{cases} \det(B) = 0 \\ B \text{ ranges over the } r \times r\text{-submatrices of } A_{\text{gen}}. \end{cases}$$

By our assumption, at least one of the $r \times r$ -submatrices of A_{gen} is non-singular. Thus Y_d is a proper closed subvariety of X defined over k .

Now suppose K/k is a field extension and $x \in X(K)'$, as before. A polynomial $\beta(t_1, \dots, t_n) = \sum_{i=1}^l \beta_i M_i \in K[t_1, \dots, t_n]$ lies in $I(W_x)[d]$ if and only if $\beta_1, \dots, \beta_l \in K$ satisfy the linear system

$$(2.4) \quad \begin{cases} \beta_1 M_1(s_\lambda(x)) + \beta_2 M_2(s_\lambda(x)) + \dots + \beta_l M_l(s_\lambda(x)) = 0 \\ \lambda \in \Lambda \end{cases}$$

with coefficients in K . The matrix A_x of this system is obtained from A_{gen} by evaluating each entry at x . (Note that it makes sense to evaluate an entry of A_{gen} at $x \in X(K)'$. Indeed,

x lies in the domain of each s_λ and every entry in A_{gen} is a monomial in the coordinates of $s_\lambda(x) \in V_x \subset \mathbb{A}_K^n$. Thus $\dim(I(W_x)[d]) = l - \text{rank}(A_x)$. If we choose x away from Y_d , then $\text{rank}(A_x) = \text{rank}(A_{\text{gen}}) = r$ and

$$\dim(I(W_x)[d]) = l - \text{rank}(A_x) = l - r = h(d) = \dim(I(V_x)[d]),$$

as desired. \square

3. THE HILBERT SCHEME

The Hilbert scheme $\text{Hilb}(n)$, constructed by A. Grothendieck [Gro95], classifies closed subvarieties of \mathbb{P}^n in the following sense. A family of subvarieties of \mathbb{P}^n parametrized by a scheme X is, by definition, a closed subvariety

$$V \subset X \times \mathbb{P}^n$$

such that the projection $V \rightarrow X$ to the first factor is flat. Families of subvarieties of \mathbb{P}^n parametrized by X are in a natural (functorial in X) bijective correspondence with morphisms $X \rightarrow \text{Hilb}(n)$. Note that $\text{Hilb}(n)$ is not a Noetherian scheme, it is only locally Noetherian. This will not present a problem for us though, because $\text{Hilb}(n)$ is a disjoint union of (infinitely many) schemes of the form $\text{Hilb}(n, p(d))$, where $p(d)$ is a Hilbert polynomial, and each $\text{Hilb}(n, p(d))$ is a projective variety defined over \mathbb{Z} . If X is an irreducible variety, and $V \rightarrow X$ is a family of subvarieties of \mathbb{P}^n , as above, then the image of the morphism $X \rightarrow \text{Hilb}(X)$ associated to this family lies in $\text{Hilb}(n, p(d))$, where $p(d)$ is the Hilbert polynomial of any fiber of $V \rightarrow X$. (Since the morphism $V \rightarrow X$ is assumed to be flat, every fiber will have the same Hilbert polynomial.)

We are now ready to proceed with the proof of Theorem 1.2. In this section we will construct a dense open subset $X_0 \subset X$, a dominant morphism $\phi: X_0 \rightarrow Z$, and a countable collection of proper k -subvarieties $Y_d \subset X_0$. We will check that each Y_d is G -invariant and defer the rest of the proof of Theorem 1.2 to the next two sections.

By our assumption X is a quasi-projective variety. In other words, X is a locally closed subvariety of some projective space \mathbb{P}_k^n . Let $V \subset X \times \mathbb{P}^n$ be the Zariski closure of the union of the graphs of $g: X \dashrightarrow \mathbb{P}^n$, as g ranges over G . Let $\pi: V \rightarrow X$ be the projection

$$(3.1) \quad \begin{array}{ccc} V := \overline{\{(x, g(x)) \mid x \in X, g \in G\}} & \subset & X \times \mathbb{P}^n \\ \pi \downarrow & & \\ X & & \end{array}$$

to the first factor. Let $X_0 \subset X$ be the flat locus of π , i.e. the largest dense open subset of X over which π is flat. (Recall that X_0 is dense in X by generic flatness.) Let $V_0 := \pi^{-1}(X_0)$. We now view $V_0 \subset X_0 \times \mathbb{P}^n$ as a family of subvarieties of \mathbb{P}^n parametrized by X_0 . By the universal property of the Hilbert scheme $\text{Hilb}(n)$, this family induces a morphism $\phi: X_0 \rightarrow \text{Hilb}(n)$. Denote the closure of the image of this morphism by Z . If K/k is a field extension and $x, y \in X_0(K)$ then by our construction

$$(3.2) \quad \phi(x) = \phi(y) \text{ if and only if } V_x = V_y.$$

Here we identify $\{x\} \times \mathbb{P}_K^n$ and $\{y\} \times \mathbb{P}_K^n$ with \mathbb{P}_K^n .

Each $g \in G$ gives rise to the (regular) section $s_g: X_0 \rightarrow V_0$ given by $x \mapsto (x, g(x))$, as g ranges over G . By the definition of V , the union of the images of these sections is dense in V_0 . Thus by Proposition 2.1 there exists a countable collection of proper k -subvarieties $Y_i \subset X_0$, such that for any field extension K/k and any $x \in X_0(K)$ away from the union of these subvarieties,

$$(3.3) \quad V_x = \overline{G \cdot x} \text{ in } \mathbb{P}_K^n.$$

Lemma 3.1. *Each subvariety $Y_d \subset X_0$ is G -invariant.*

Proof. We revisit the construction of Y_d in the proof of Proposition 2.1. For the first reduction at the beginning of that proof, note that V is, by definition, a subvariety of $X \times \mathbb{P}^n$. Thus we may choose an affine subspace $\mathbb{A}^n \subset \mathbb{P}^n$, so that X intersects \mathbb{A}^n non-trivially and replace V by $V \cap (X \times \mathbb{A}^n)$. The second reduction is carried out by passing from X to X_0 .

With these two reductions in place, $Y_d \subsetneq X_0$ was constructed as follows. Let t_1, \dots, t_n be the affine coordinates on \mathbb{A}^n and M_1, \dots, M_l be the distinct monomials of degree d in t_1, \dots, t_n . Let

$$A_{\text{gen}} := \begin{pmatrix} M_1(g) & M_2(g) & \dots & M_l(g) \\ g \in G \end{pmatrix},$$

where we view g as a rational function $X \dashrightarrow X \cap \mathbb{A}^n \subset \mathbb{A}^n$. Let r be the rank of A_{gen} . The subvariety $Y_d \subset X$ is then defined by the condition that $\text{rank}(A_{\text{gen}}) < r$. That is, Y_d is the intersection of the closed subvarieties of X_0 cut out by the system of equations

$$\begin{cases} \det(B_{i_1, \dots, i_r, g_1, \dots, g_r}) = 0 \\ 1 \leq i_1 < \dots < i_r \leq l \text{ and } g_1, \dots, g_r \text{ are distinct elements of } G. \end{cases}$$

Here $B_{i_1, \dots, i_r, g_1, \dots, g_r}$ is the $r \times r$ -submatrix of A_{gen} consisting of columns i_1, \dots, i_r and rows indexed by g_1, \dots, g_r .

To prove the lemma we need to check that if a point $x \in X_0(K)$ lies on Y_d then so does $g(x)$ whenever $g(x)$ is defined. Here K/k is a field extension. Indeed,

$$B_{i_1, \dots, i_r, g_1, \dots, g_r}(gx) = B_{i_1, \dots, i_r, g_1g, \dots, g_rg}(x).$$

If $x \in Y_d$ then the matrix on the right hand side is singular for any choice of $1 \leq i_1, \dots, i_r \leq l$ and $g_1, \dots, g_r \in G$, as desired. \square

4. CONCLUSION OF THE PROOF OF THEOREM 1.2(A), (B), (D) AND (E)

(b) By (3.2), $\phi(x) = \phi(y)$ if and only if $V_x = V_y$ in \mathbb{P}_K^n . By (3.3), $V_x = \overline{G \cdot x}$, $V_y = \overline{G \cdot y}$, where the closure is taken in \mathbb{P}_K^n . This shows that $\phi(x) = \phi(y)$ if and only if $G \cdot x$ and $G \cdot y \subset X_K$ have the same closure in \mathbb{P}_K^n . On the other hand, $G \cdot x$ and $G \cdot y$ have the same closure in \mathbb{P}_K^n if and only if they have the same closure in X_K .

(a) It suffices to show that the rational maps $\phi \circ g$ and $\phi: X \dashrightarrow \text{Hilb}(n)$ agree on the generic point x_{gen} of X for every $g \in G$. Choose $g \in G$ and fix it for the rest of the proof. Then $x := x_{\text{gen}}$ and $y := g(x_{\text{gen}})$ are K_{gen} -points of X , where $K_{\text{gen}} := k(X)$. Since G is dominant, neither x nor y lie on any proper subvariety of X defined over k . In particular, they do not lie in the indeterminacy locus of any $h \in G$ or on Y_i for any $i \geq 1$. By (3.2), proving that $\phi(x) = \phi(y)$ is equivalent to showing that

$$(4.1) \quad V_x = V_y \text{ in } \mathbb{P}_{K_{\text{gen}}}^n,$$

where $V_x, V_y \in \mathbb{P}_{K_{\text{gen}}}^n$ are the fibers of x and y , under $\phi: V \rightarrow X$ in $\mathbb{P}_{K_{\text{gen}}}^n$.

Since $x, y \in X_0(K_{\text{gen}})$ do not lie on Y_i for any $i \geq 1$, part (b) tells us that $V_x = \overline{G \cdot x}$ and $V_y = \overline{G \cdot y}$. If g is invertible in G , this immediately implies (4.1), since $G \cdot x = G \cdot y$. In general (if g is not necessarily invertible), we use the following alternative argument to prove (4.1). Since $y = g(x)$, we have $G \cdot y \subset G \cdot x$ and thus

$$(4.2) \quad V_y \subset V_x.$$

Moreover, since $\pi: V \rightarrow X$ is flat over X_0 , V_x and V_y have the same Hilbert function. Now (4.1) follows from (4.2).

(d) If g is an invertible element of G then one easily checks that the variety $V \subset X \times \mathbb{P}^n$ defined in (3.1), is g -invariant, where g acts on $X \times \mathbb{P}^n$ via the first factor. Consequently, the flat locus $X_0 \subset X$ of the projection $\pi: V \rightarrow X$ is g -invariant.

(e) By part (a), $\phi((G \cdot x) \cap X_0) = \phi(x)$ and thus

$$(4.3) \quad \overline{(G \cdot x) \cap X_0} \subset \phi^{-1}(\phi(x)),$$

where the closure is taken in X_0 . On the other hand, if $y \in X_0$ and $\phi(y) = \phi(x)$ then by (3.2), $V_y = V_x$. Since G is a monoid, $y \in V_y$ and thus $y \in V_x$. In other words, $\phi^{-1}(\phi(x)) \cap X_0 \subset V_x \cap X_0$. Combining this with (4.3), we obtain

$$(4.4) \quad \overline{(G \cdot x) \cap X_0} \subset \phi^{-1}(\phi(x)) \subset V_x \cap X_0.$$

On the other hand, by (3.3), $G \cdot x$ is dense in V_x . Thus $(G \cdot x) \cap X_0$ is dense in $V_x \cap X_0$, i.e.,

$$\overline{(G \cdot x) \cap X_0} = V_x \cap X_0.$$

We conclude that both containments in (4.4) are equalities. In particular, $\overline{(G \cdot x) \cap X_0} = \phi^{-1}(\phi(x))$, as desired.

5. PROOF OF THEOREM 1.2(C)

By part (a), $\phi^*(k(Z)) \subset k(X)^G$. Let Y be a k -variety whose function field $k(Y)$ is $k(X)^G$. The inclusions $k(Z) \xrightarrow{i^*} k(Y) = k(X)^G \xrightarrow{\psi^*} k(X) = k(X_0)$ induce dominant rational maps

$$\begin{array}{ccc} & X_0 & \\ & \downarrow \psi & \\ \phi \swarrow & Y & \searrow i \\ & Z & \end{array}$$

After replacing X_0 , Z , and Y by suitable dense open subvarieties, we may assume that all three maps in the above diagram are regular.

Let K/k be a field and $x, y \in X_0(K)$ be as in Theorem 1.2(b). We claim that if $\phi(x) = \phi(y)$ then $\psi(x) = \psi(y)$. Indeed, by Theorem 1.2(b), $\phi(x) = \phi(y)$ implies that $\overline{G \cdot x} = \overline{G \cdot y}$ in X_K . Then

$$(5.1) \quad \overline{(G \cdot x) \cap X_0} = \overline{(G \cdot y) \cap X_0} \text{ in } X_0.$$

By our construction, ψ is G -equivariant, where G acts trivially on Y . Thus ψ sends all of $\overline{(G \cdot x) \cap X_0}$ to the point $\psi(x)$ and all of $\overline{(G \cdot y) \cap X_0}$ to the point $\psi(y)$ in Y . By (5.1), $\psi(x) = \psi(y)$, as claimed.

In particular, if K/k is field extension and $x, y: \text{Spec}(K) \rightarrow X$ are dominant points, then x and y satisfy the conditions of Theorem 1.2(b) and thus

$$\phi(x) = \phi(y) \text{ if and only if } \psi(x) = \psi(y).$$

By Lemma 5.1 below, i is purely inseparable. This proves the first assertion of Theorem 1.2(d).

To prove the second assertion of part (d), we simply replace Z by Y and ϕ by ψ . Since i is inseparable, properties of ϕ asserted by Theorem 1.2 are shared by ψ .

Lemma 5.1. *Let $\phi: X \xrightarrow{\psi} Y \xrightarrow{i} Z$ be dominant maps of irreducible k -varieties. Suppose that for any pair of dominant points $x, x': \operatorname{Spec}(K) \rightarrow X$, where K/k is a field extension,*

$$(5.2) \quad \phi(x) = \phi(x') \text{ if and only if } \psi(x) = \psi(x').$$

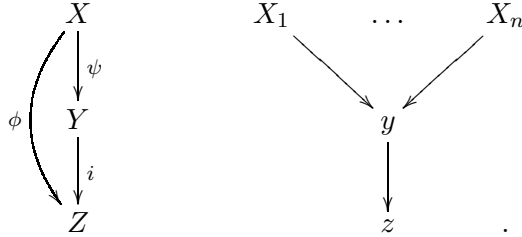
Then the field extension $k(Y)/i^(k(Z))$ is purely inseparable.*

Proof. Let F be the algebraic closure of $k(X)$ and $x: \operatorname{Spec}(F) \rightarrow X$ be the dominant F -point of X obtained by composing the natural projection $\operatorname{Spec}(F) \rightarrow \operatorname{Spec}(k(X))$ with the generic point $\operatorname{Spec}(k(X)) \rightarrow X$. Set $z := \phi(x)$. The fiber $\phi^{-1}(z)$ is an F -subvariety of X_F . Denote its irreducible components by X_1, \dots, X_n .

By the fiber dimension theorem, the generic point $x_i: \operatorname{Spec}(F(X_i)) \rightarrow X_i \hookrightarrow X$ is dominant for every $i = 1, \dots, n$. If K/F is a compositum of $F(X_1), \dots, F(X_n)$ over F and $(x_i)_K$ is the composition of the projection $\operatorname{Spec}(K) \rightarrow \operatorname{Spec}(F(X_i))$ with x_i , then

$$\phi((x_1)_K) = \dots = \phi((x_n)_K) = z_K = \phi(x_K).$$

Our assumption (5.2) now tells us that $\psi((x_1)_K) = \dots = \psi((x_n)_K) = \psi(x_K)$. Since x is, by definition, an F -point of X , we see that $\psi((x_1)_K) = \dots = \psi((x_n)_K) = \psi(x)$ descends to a dominant F -point $y: \operatorname{Spec}(F) \rightarrow Y$, where $i(y) = z$. In other words ψ maps each X_i to the single point $y \in Y(F)$, as depicted in the following diagram:



Thus $\psi(\phi^{-1}(z)) = y$. Equivalently, $\psi(\psi^{-1}(i^{-1}(z))) = y$ or $i^{-1}(z) = y$. Applying the fiber dimension theorem one more time, we see that $\dim(Y) = \dim(Z)$. Since $z: \operatorname{Spec}(F) \rightarrow Z$ is dominant, and F is algebraically closed, the number of preimages of z in Y equals the separability degree of $k(Y)$ over $i^*(k(Z))$. In our case the preimage of z is a single point $y \in Y(F)$; hence, $k(Y)$ is purely inseparable over $i^*(k(Z))$. This completes the proof of Lemma 5.1 and thus of Theorem 1.2. \square

Remark 5.2. We do not know whether or not $\phi^*k(Z)$ always coincides with $k(X)^G$, where Z is the closure of the image of the rational map $\phi: X \dashrightarrow \operatorname{Hilb}(n)$ we constructed. As we have just seen, this is always the case in characteristic zero, so the question is only of interest in prime characteristic. In the setting, where an algebraic group is acting rationally on X (which is a bit more general than that of Theorem 1.1), an analogous question concerning the map $\phi: X \dashrightarrow C$ into an appropriate Chow variety constructed by Rosenlicht, was left open in [Ros56]. A. Seidenberg [Sei79] subsequently showed that, indeed, $k(C) = k(X)^G$ under the assumption that the G -action on X is regular (as in Theorem 1.1).

6. SOME COROLLARIES AND EXAMPLES

The following corollary of Theorem 1.2 is a generalization of the Medvedev-Scanlon conjecture [MS14, Conjecture 7.14] in the case, where the base field k is uncountable.

Corollary 6.1. *Let k be an uncountable algebraically closed field, X be an irreducible quasi-projective k -variety, and G be a semigroup of dominant regular maps $X \rightarrow X$. If $k(X)^G = k$ then $G \cdot x$ is dense in X for some $x \in X(k)$.*

Proof. Let $X_0 \subset X$ be a dense open subset, and $\phi: X_0 \rightarrow Z$ be a morphism such that $\phi^*k(Z) = k(X)^G$, as in Theorem 1.2. Since $k(X)^G = k$, the variety Z is a single point.

Let $S \subset X_0(K)$ be the set of k -points of X_0 away from the exceptional sets Y_i for every $i \geq 1$. Since k is algebraically closed and uncountable, S is Zariski dense in X_0 . By Theorem 1.2(b),

$$(6.1) \quad Y = \overline{G \cdot x} \text{ is independent of the choice of } x \in S.$$

Since $g: X \rightarrow X$ is dominant for every $g \in G$, we see that the union

$$\bigcup_{x \in S} \overline{G \cdot x}$$

is dense in X . On the other hand, by (6.1), this union equals Y , which is closed in X . We conclude that $Y = X$, i.e., $\overline{G \cdot x} = X$ for every $x \in S$. \square

Note that in the statement of Corollary 6.1 we do not make any assumptions of the semigroup G ; in particular, it may be uncountable. However, the following corollary shows that G -orbit closures in very general position are controlled by a countable subsemigroup $H \subset G$.

Corollary 6.2. *Under the assumptions of Theorem 1.2, with $k(Z) = k(X)^G$, there exists a countable subsemigroup H of G such that*

(a) $k(X)^G = k(X)^H$, and

(b) *Moreover, there exists a countable collection $\{W_i \mid i \geq 1\}$ of H -invariant subvarieties of X_0 defined over k with the following property. For any field extension F/k and any $x \in X_0(F)$ away from the indeterminacy loci of all $g \in G$ and away from $\cup_{i \geq 1} W_i$, we have $\overline{G \cdot x} = \overline{H \cdot x}$.*

Proof. Our proof will rely on the following elementary

Claim: Let W be an algebraic variety (not necessarily irreducible), and S be a dense collection of points in W . Then there exists a countable subcollection $S' \subset S$ which is dense in W .

After replacing W by a dense open subset, and removing the points of S that do not lie in this dense open subset, we may assume without loss of generality that $W \subset \mathbb{A}^n$ is affine. Let $I(W)$ be the ideal of W in $k[x_1, \dots, x_n]$ and $I(W)[d]$ be the finite-dimensional vector space of polynomials of degree $\leq d$ vanishing on W . It is easy to see that for each d there is a finite subset $S_d \subset S$ such that $I(S_d)[d] = I(W)[d]$. Taking $S' = \cup_{d=1}^{\infty} S_d$, we see that $I(S')[d] = I(W)[d]$ for every $d \geq 1$. Thus $I(S') = I(W)$ and S' is dense in W . This proves the claim.

We now proceed with the proof of Corollary 6.2. We will assume that $X \subset \mathbb{P}^n$ is a locally closed subvariety of \mathbb{P}^n for some $n \geq 1$. Let $V \subset X \times \mathbb{P}^n$ be the closure of the union of the images of

$$s_g: X \dashrightarrow X \times \mathbb{P}^n,$$

as in (3.1). Applying the claim to the generic points of the closures of the images of s_g , as $g \in G$, we see that there exists a countable collection of elements $\{h_i \mid i \geq 1\}$ such that the images of s_{h_i} are dense in V . Let H be the countable subsemigroup of G generated by these h_i . Denote the flat locus of the projection $\pi: V \rightarrow X$ by $X_0 \subset X$ and the morphism associated to π , viewed as a family of subvarieties over X_0 , by $\phi: X_0 \rightarrow \text{Hilb}(n)$. Arguing as in Section 3, we see that there exists a countable collection $\{W_i \mid i \geq 1\}$ of H -invariant closed subvarieties of X_0 such that for any field extension K/k and any point $x \in X_0(K)$ away from each W_i and from the indeterminacy locus of every $g \in G$, $V_x = \overline{H \cdot x}$ in X ; cf. (3.3). Since $H \cdot x \subset G \cdot x \subset V_x$, we have $\overline{H \cdot x} = \overline{G \cdot x} = V_x$. This proves part (b).

To prove part (a), note that by our construction, $\phi^*k(Z) \subset k(X)^G \subset k(X)^H$, and by Theorem 1.2(c), $k(X)^H$ is purely inseparable over $\phi^*k(Z)$. Thus $k(X)^H$ is purely inseparable over $k(X)^G$.

It remains to show that, in fact, $k(X)^H = k(X)^G$. Choose $a \in k(X)^H$. Then a satisfies some purely inseparable polynomial $p(t) \in k(X)^G[t]$. For any $g \in G$, $g(a)$ also satisfies $p(t)$. Since a is the only root of $p(t)$ in $k(X)$, we conclude that $g(a) = a$. In other words, $a \in k(X)^G$, i.e., $k(X)^H = k(X)^G$, as desired. \square

Example 6.3. In general one cannot expect to find a finitely generated subsemigroup $H \subset G$ such that $k(X)^G = k(X)^H$. For example, let X be a complex abelian variety of dimension ≥ 1 , and G be the group of translations on X by torsion points of $X(\mathbb{C})$. Then G is countable and $\mathbb{C}(X)^G = \mathbb{C}$. On the other hand, any finitely generated subgroup H of G is finite and $\mathbb{C}(X)^H$ is a finite subextension of $\mathbb{C}(X)$. \square

The following examples show that the countable collection $\{Y_i, i \geq 1\}$ of proper “exceptional” subvarieties of X cannot be replaced by a finite collection in the statement of Theorem 1.2.

Example 6.4. Let E be an elliptic curve over k , $X = E \times E$, σ be an automorphism of X given by $\sigma(x, y) := (x, x + y)$, and $G \xrightarrow{\sim} \mathbb{N}$ (or $G \xrightarrow{\sim} \mathbb{Z}$) is generated by σ as a semigroup (or as a group). In this case Z and ϕ are unique (up to birational isomorphism), and is easy to see that ϕ is just projection to the first factor, $\phi: X \rightarrow Z := E$. The fiber $X_z = \{x\} \times E$ is the closure of a single orbit if and only if x is of infinite order in E . Thus the “exceptional set” $Y_1 \cup Y_2 \cup \dots$ has to contain countably many “vertical” curves $\{x\} \times E$, as x ranges over the torsion points of E .

Example 6.5. Let $X = \mathbb{P}^n$ and $G \xrightarrow{\sim} \mathbb{N}$ be the cyclic semigroup generated by a single dominant morphism $\sigma: X \rightarrow X$ of degree ≥ 2 . Assume that k is algebraically closed. Then the exceptional collection Y_1, Y_2, \dots is dense in X ; in particular, it cannot be finite.

Proof. We claim that $\text{trdeg}_k k(X)^G < n$. Indeed, assume the contrary. Then the field extension $k(X)/k(X)^G$ is algebraic and finitely generated; hence, it is finite. Now we can view $\sigma^*: k(X) \rightarrow k(X)$ as a $k(X)^G$ -linear transformation of a finite-dimensional $k(X)^G$ -vector space $k(X)$. Since σ is injective, we conclude that it is also surjective, and thus $\sigma: X \rightarrow X$ has degree 1, contradicting our choice of σ .

The claim tells us that the general fiber (and hence, every non-empty fiber) of the map $\sigma: X \rightarrow Z$ has dimension ≥ 1 . Suppose $x \in X(k)$ is a periodic point of X , i.e., $\sigma^n(x) = x$ for some $n \geq 1$. Then $G \cdot x$ is finite, and thus $\overline{G \cdot x}$ is 0-dimensional. Consequently, $G \cdot x$ cannot be dense in the fiber $\phi^{-1}(\phi(x))$, so x has to lie in the exceptional locus $Y_1 \cup Y_2 \cup \dots$. On the other hand, by a result of N. Fakhruddin [Fak03, Corollary 5.3] periodic k -points for σ are dense in $X = \mathbb{P}^n$. \square

7. ROSENLICHT’S THEOREM REVISITED

To put our proof of Theorem 1.2 in perspective, we will now show how our arguments can be modified to obtain a proof of Theorem 1.1 in the case, where $X \subset \mathbb{P}^n$ is an irreducible quasi-projective k -variety.

An action of an algebraic group G on X given by a k -morphism

$$\begin{aligned} \psi: G \times X &\rightarrow X \times X \subset X \times \mathbb{P}^n \\ (g, x) &\mapsto (x, g(x)). \end{aligned}$$

Once again, we define $V \subset X \times \mathbb{P}^n$ to be the closure of the image of ψ , $\pi: V \rightarrow X$ to be projection to the first factor, and $V_x := \pi^{-1}(x)$ to be the fiber of $x \in X$, as in (3.1). The role of Proposition 2.1 will now be played by Lemma 7.1 below. Note that the countable collection of exceptional subvarieties $Y_i \subset X$ from Proposition 2.1 does not come up, and the proof is considerably simpler in this setting.

Lemma 7.1. *There exists a G -invariant dense open subvariety $U \subset X$ with the following property: for any field K/k and any $x \in U(K)$, $G \cdot x$ is dense in V_x .*

Proof. First we will prove the lemma under the assumption that G is irreducible. In this case $V = \overline{\psi(G \times X)}$ is also irreducible. By Chevalley's theorem, $\psi(G \times X)$ contains a dense open subvariety W of V . After replacing W by the union of its g -translates, as g ranges over G , we may assume that W is G -invariant. Here G acts on V via the second factor. In particular, $W \cap V_x = \{x\} \times (G \cdot x)$. Set $C := V - W$ and

$$D := \{x \in X \mid C \text{ contains an irreducible component of } V_x\}.$$

One readily checks that D is a G -invariant closed subvariety of X . We claim that $D \neq X$. Indeed, by the fiber dimension theorem every irreducible component of V_x has dimension $\geq \dim(V) - \dim(X)$. If $D = X$, then $\dim(C) = \dim(V)$ and thus $C = V$, since V is irreducible. Consequently, $W = \emptyset$, a contradiction. This proves the claim. We conclude that the dense open subset $U := X - D$ of X has the desired property. This completes the proof of the lemma under the assumption that G is irreducible.

In general denote the irreducible components of G by G^0, G^1, \dots, G^r , where G^0 is the identity component. Let V^0 be the closure of the image of the action map $G^0 \times X \rightarrow X \times X$ in $X \times \mathbb{P}^n$. Then

$$V = V^0 \cup (G^1 \cdot V^0) \cup \dots \cup (G^r \cdot V^0).$$

As we have just shown, there exists a G^0 -invariant dense open subset $U \subset X$ such that $G^0 \cdot x$ is dense in V_x^0 for any $x \in U$. Then clearly $G \cdot x$ is dense in V_x . A priori U is only G^0 -invariant, but we can make it G -invariant by replacing it with the dense open subset $U^0 \cap (G^1 \cdot U^0) \cap \dots \cap (G^r \cdot U^0)$. \square

From now on we will replace X by U , and thus assume that $G \cdot x$ is dense in V_x for every $x \in X$. Once again, we define X_0 as the flat locus of the projection $\pi: V \rightarrow X$. The same argument as in the proof of Theorem 1.2(d) shows that X_0 is G -invariant. Once again, we define $\phi: X_0 \rightarrow \text{Hilb}(n)$ as the morphism associated to $\pi^{-1}(X_0) \subset X \times \mathbb{P}^n$, which we view as a family of subvarieties of \mathbb{P}^n parametrized by X_0 . By construction ϕ separates the fibers of π ; these fibers are G -orbit closures by Lemma 7.1. To complete the proof of Theorem 1.1 it remains to show that ϕ actually separates the orbits in X_0 , and not just their closures. Note that since separating orbit closures is the most we can do in the dynamical setting, Lemma 7.2 has no counterpart in our proof of Theorem 1.2.

Lemma 7.2. *Let K/k be a field extension and $x \in X_0(K)$ be a K -point. Then the orbit $G \cdot x$ is closed in $(X_0)_K$.*

Proof. After base-changing to the algebraic closure \overline{K} , we may assume that $K = k$ is algebraically closed. Since the morphism $\pi: V \rightarrow X$ is flat over X_0 , $\dim(V_x)$ does not depend on the choice of x in $X_0(k)$. Thus the dense open subset $G \cdot x$ of V_x also has the same dimension for every $x \in X_0(k)$.

Assume the contrary: $G \cdot x$ is not closed in X_0 for some $x \in X_0(k)$. Then the complement to $G \cdot x$ in $\overline{G \cdot x}$ has a k -point $y \in X_0(k)$. Since $G \cdot x$ is open and dense in $\overline{G \cdot x}$, we have $\dim(G \cdot y) < \dim(G \cdot x)$, a contradiction. \square

This completes the proof of Theorem 1.1(a). Part (b) is proved in exactly the same way as Theorem 1.2(c) in Section 5.

ACKNOWLEDGEMENTS. The authors are grateful to E. Amerik and T. Tucker for helpful correspondence.

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